

Superconformal symmetry in the interacting theory of $(2,0)$ tensor multiplets and self-dual strings

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Abstract: We investigate the concept of superconformal symmetry in six dimensions, applied to the interacting theory of $(2,0)$ tensor multiplets and self-dual strings. The action of a superconformal transformation on the superspace coordinates is found, both from a six-dimensional perspective and by using a superspace with eight bosonic and four fermionic dimensions. The transformation laws for all fields in the theory are derived, as well as general expressions for the transformation of on-shell superfields. Superconformal invariance is shown for the interaction of a self-dual string with a background consisting of on-shell tensor multiplet fields, and we also find an interesting relationship between the requirements of superconformal invariance and those of a local fermionic κ -symmetry. Finally, we try to construct a superspace analogue of the Poincaré dual to the string world-sheet and consider its properties under superconformal transformations.

Contents

1	Introduction	2
2	Conformal invariance in the bosonic theory	4
2.1	Conformal coordinate transformations	4
2.2	Conformal field transformations	7
2.3	The bosonic tensor multiplet	8
3	Conformal invariance in the supersymmetric theory	10
3.1	Including spinors in the model	10
3.2	Superconformal coordinate transformations	12
3.3	Superfields	18
3.4	The supersymmetric interaction terms	22
4	The Poincaré dual to the string world-sheet	23
4.1	The bosonic model	24
4.2	The superconformal model	26
	References	30

1 Introduction

One of the most interesting discoveries in string/ M -theory during the past decade is without doubt the six-dimensional $(2,0)$ theories, named after the supersymmetry algebra under which they are invariant [1]. These are superconformal quantum theories without dynamical gravity, and first appeared in a compactification of Type IIB string theory on a four-dimensional hyper-Kähler manifold [2]. They obey an ADE classification (see [3] for an intrinsically six-dimensional motivation for why the simply laced Lie algebra series A , D and E appear) but have no other discrete or continuous parameters. However, the theories have a moduli space parametrized by the expectation values of a set of scalar fields.

There is a second origin for the A -series of these theories in terms of M -theory, where they arise as the world-volume theory on a stack of parallel $M5$ -branes [4, 5]. More specifically, a stack of $r+1$ branes yields the A_r version of $(2,0)$ theory, where r denotes the rank of the associated Lie algebra. The fluctuations of the $M5$ -branes are described by r so called $(2,0)$ tensor multiplets and the $5r$ moduli correspond to the transverse distances between the branes. It is also possible for membranes to stretch between two $M5$ -branes [6–8]; the intersections will then appear as self-dual strings from the six-dimensional world-volume perspective on the $M5$ -brane. We get in total $r(r+1)/2$ different species of such strings, corresponding to the number of ways to connect the branes. The string tension is proportional to the distance between the branes in question, and is therefore related to the moduli of the theory. Specifically, if the branes coincide, the strings become tensionless. This picture yields in a simple

way the different degrees of freedom of $(2, 0)$ theory; for a more complete discussion, see e.g. [9].

An intrinsically six-dimensional formulation of the $(2, 0)$ theories is still lacking, and it is the ultimate goal of our research to find such a definition. In our previous work, we have pursued a program where we consider the theory at a point away from the origin of its moduli space, where the strings are tensile. This introduces a scale in the theory and breaks the conformal invariance spontaneously, but also provides a basis for doing perturbation theory, since in the limit of large string tension, the theory describes the well understood free $(2, 0)$ tensor multiplet and free self-dual strings. The dimensionless expansion parameter will then be the square of some typical energy divided by the string tension. For simplicity, we have chosen to work in the A_1 version of $(2, 0)$ theory, which includes a single type of string and a single tensor multiplet.

In this paper, we consider the hitherto uninvestigated superconformal symmetry of our model. It is well known that superconformal field theories cannot exist in space-times with more than six dimensions. Moreover, in six dimensions, the largest possible supersymmetry consistent with superconformal invariance is $\mathcal{N} = (2, 0)$. This means that $(2, 0)$ theory, regarded as a superconformal theory, has the largest possible supersymmetry in the highest possible dimension. This observation alone provides a motivation for studying these theories.

The aim of the present paper is firstly to provide a framework for the study of superconformal invariance in six-dimensional $(2, 0)$ theory by using superfields. Secondly, we want to investigate whether our model [10] for a self-dual string interacting with a tensor multiplet background is superconformally invariant, and in what sense. Finally, we would like to use the new superconformal tools to find some clues on how to formulate the full interacting theory, i.e. when the tensor multiplet fields do not obey their free equations of motion.

It is our intention to make the paper self-contained, thereby reviewing and summarizing some well-known results on conformal and superconformal symmetry. We think that it is worthwhile to include these, in order to understand how the superfield transformations work and how they are derived. It is also, in some cases, non-trivial to adapt the known results to our notations and conventions.

The outline of this paper is as follows: Section 2 is devoted to the study of (bosonic) conformal transformations in six dimensions, as a warm-up exercise preceding the superconformal model. We find explicit expressions for the action of the conformal group on the space-time coordinates, both from a six-dimensional perspective and by considering a projective hypercone in eight dimensions, where the conformal transformations act linearly. We also study how different fields behave under conformal transformations, especially the ones in the bosonic sector of the $(2, 0)$ tensor multiplet. The results of Section 2 are not new, but still useful from a pedagogical point of view.

Section 3 discusses superconformal symmetry. As in the previous section, we first find how the transformations act on coordinates, this time in a superspace with six bosonic and sixteen fermionic dimensions. The coordinate transformations are also found by considering a projective supercone in a higher-dimensional superspace; a

method which, to the best of our knowledge, has not appeared previously in the literature. Next, we use a superfield formalism as a way of compactifying and simplifying the notation and derive how superconformal transformations act on the superfields of the complete $(2, 0)$ tensor multiplet and on self-dual spinning strings. The explicit transformation laws are used to show that the model describing a string interacting with a tensor multiplet background is superconformally invariant. We find the new and non-trivial result that the requirements of superconformal symmetry and those of a local fermionic κ -symmetry impose similar restrictions on the possible coupling terms, thereby indicating the uniqueness of the theory.

Finally, Section 4 discusses the construction of a superspace analogue of the Poincaré dual to the string world-sheet and its properties under superconformal transformations.

2 Conformal invariance in the bosonic theory

Before discussing the issue of superconformal invariance, it is worthwhile to consider a simpler case, which still contains some important aspects of the problem. Therefore, we begin by working with the model describing a spinless self-dual string interacting with the bosonic part of the $(2, 0)$ tensor multiplet in six dimensions [9]. The results in this section have appeared previously in the literature, but it is nevertheless useful to review some aspects in order to get a complete picture. We will also refer (for comparison) to formulae in this section later in this paper, to point out similarities and differences between the bosonic and the superconformal models.

2.1 Conformal coordinate transformations

In this subsection, we consider the action of conformal transformations on the coordinates x^μ , $\mu = 0, \dots, 5$, in a six-dimensional space-time. These transformations are called *passive*, meaning that they act on the coordinates themselves, rather than on the fields of the theory.

Conformal transformations act on space-time in such a way that the *angle* between two intersecting curves is left invariant. This means that the infinitesimal proper time interval

$$d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu \quad (2.1)$$

transforms according to

$$d\tau^2 \rightarrow \Omega^2(x)d\tau^2, \quad (2.2)$$

where $\eta_{\mu\nu}$ is the flat space-time metric (with "mostly plus" signature) and $\Omega(x)$ denotes a space-time dependent quantity that also involves the parameters of the conformal transformation. For an infinitesimal transformation, such that $\Omega(x) = 1 + \Lambda(x)$, this yields that

$$\delta(d\tau^2) = 2\Lambda(x)d\tau^2, \quad (2.3)$$

where $\Lambda(x)$ is an infinitesimal function.

It is well known that the most general infinitesimal coordinate transformation that respects Eq. (2.3) is given by

$$\delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \lambda x^\mu + c^\mu x^2 - 2c \cdot x x^\mu, \quad (2.4)$$

where $c \cdot x \equiv \eta_{\mu\nu} c^\mu x^\nu = c_\nu x^\nu$. The (constant) parameters a^μ , $\omega^{\mu\nu}$, λ and c^μ are related to the different transformations of the conformal group according to the following table:

symmetry	generator	parameter
translations	$P_\mu = \partial_\mu$	a^μ
rotations and Lorentz boosts	$M_{\mu\nu} = x_{[\nu} \partial_{\mu]}$	$\omega^{\mu\nu}$
dilatations	$D = x^\mu \partial_\mu$	λ
special conformal transformations	$K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$	c^μ

where the differential expressions for the generators make the equation

$$\delta x^\mu = (a^\nu P_\nu + \omega^{\nu\rho} M_{\nu\rho} + \lambda D + c^\nu K_\nu) x^\mu \quad (2.5)$$

valid. In total, we have 28 parameters, in agreement with the dimensionality of the conformal group in six dimensions, denoted by $SO(6, 2)$.

From Eq. (2.4), it is easily shown that the differential dx^μ transforms according to

$$\delta(dx^\mu) = \left(\omega^{\mu\nu} + 4c^{[\mu} x^{\nu]} \right) dx_\nu + (\lambda - 2c \cdot x) dx^\mu, \quad (2.6)$$

which implies that the proper time interval indeed transforms according to Eq. (2.3) with $\Lambda(x) = \lambda - 2c \cdot x$.

Leaving the coordinate transformations aside for a while, we turn to the commutation relations relating the generators of the conformal group to each other. By direct calculation, it is easily verified that the differential operators defined in the table above obey

$$\begin{aligned} [P_\mu, M_{\nu\rho}] &= -\eta_{\mu[\nu} P_{\rho]} & [P_\mu, D] &= P_\mu \\ [K_\mu, M_{\nu\rho}] &= -\eta_{\mu[\nu} K_{\rho]} & [K_\mu, D] &= -K_\mu \\ [P_\mu, K_\nu] &= -4M_{\mu\nu} - 2\eta_{\mu\nu} D & [M_{\mu\nu}, D] &= 0 \\ [M_{\mu\nu}, M^{\rho\sigma}] &= 2\delta_{[\mu}^{[\rho} M_{\nu]}^{\sigma]}. \end{aligned} \quad (2.7)$$

This defines the conformal group $SO(6, 2)$ in six dimensions. The $SO(6, 2)$ structure may be made more explicit by introducing a new set of generators $J_{\hat{\mu}\hat{\nu}}$, $\hat{\mu}, \hat{\nu} = 0, \dots, 7$, according to

$$\begin{aligned} M_{\mu\nu} &= J_{\mu\nu} \\ P_\mu &= 2(J_{7\mu} + J_{6\mu}) \\ K_\mu &= 2(J_{7\mu} - J_{6\mu}) \\ D &= 2J_{76}. \end{aligned} \quad (2.8)$$

These obey the Lorentz-type commutation relations

$$[J_{\hat{\mu}\hat{\nu}}, J_{\hat{\rho}\hat{\sigma}}] = \eta_{\hat{\rho}[\hat{\mu}} J_{\hat{\nu}]\hat{\sigma}} - \eta_{\hat{\sigma}[\hat{\mu}} J_{\hat{\nu}]\hat{\rho}}, \quad (2.9)$$

where the metric $\eta_{\hat{\mu}\hat{\nu}} = \text{diag}(-1, 1, 1, 1, 1, 1, 1, -1)$. Thus, the conformal transformations act as rotations in an eight-dimensional space with two time-like directions.

There is a formulation (first suggested by Dirac [11], whose work was later continued by Kastrup [12] and by Mack and Salam [13]) in which a d -dimensional space-time is regarded as a projective hypercone in a space with one extra space-like and one extra time-like dimension, just like the one suggested in the previous paragraph. In the higher-dimensional space, conformal symmetry acts linearly, i.e. as a rotation with the generator $J_{\hat{\mu}\hat{\nu}}$. Denote the coordinates in this space by $y^{\hat{\mu}}$, and define the projective hypercone by

$$y^2 = \eta_{\hat{\mu}\hat{\nu}} y^{\hat{\mu}} y^{\hat{\nu}} = 0. \quad (2.10)$$

This equation is clearly Lorentz invariant in the eight-dimensional space, and therefore conformally invariant from a six-dimensional point of view.

Next, let the infinitesimal rotation parameters be assembled in the matrix $\pi^{\hat{\mu}\hat{\nu}}$, such that

$$\delta y^{\hat{\mu}} = \pi^{\hat{\rho}\hat{\sigma}} J_{\hat{\rho}\hat{\sigma}} y^{\hat{\mu}} = \pi^{\hat{\mu}\hat{\sigma}} y_{\hat{\sigma}}. \quad (2.11)$$

The last equality follows from the obvious definition

$$J_{\hat{\mu}\hat{\nu}} = y_{[\hat{\nu}} \partial_{\hat{\mu}}], \quad (2.12)$$

where the derivative acts on the y -variables.

To make contact with the six-dimensional space-time, we impose that

$$\pi^{\hat{\rho}\hat{\sigma}} J_{\hat{\rho}\hat{\sigma}} = \omega^{\rho\sigma} M_{\rho\sigma} + a^\rho P_\rho + c^\rho K_\rho + \lambda D, \quad (2.13)$$

which when combined with Eq. (2.8) yields that

$$\begin{aligned} \omega^{\mu\nu} &= \pi^{\mu\nu} \\ a^\mu &= \frac{1}{2} (\pi^{7\mu} + \pi^{6\mu}) \\ c^\mu &= \frac{1}{2} (\pi^{7\mu} - \pi^{6\mu}) \\ \lambda &= \pi^{76}. \end{aligned} \quad (2.14)$$

Next, we parametrize the projective hypercone by

$$\begin{aligned} y^\mu &= \gamma x^\mu \\ y^7 - y^6 &= \gamma \\ y^7 + y^6 &= \gamma \eta_{\rho\sigma} x^\rho x^\sigma, \end{aligned} \quad (2.15)$$

where γ is related to the projectiveness of the hypercone and therefore $\gamma \neq 0$.

Under a conformal transformation as in Eq. (2.11), it turns out that the quantity x^μ introduced in Eq. (2.15) must transform exactly as the coordinates x^μ in Eq. (2.4), while γ transforms according to

$$\delta\gamma = (2\eta_{\mu\nu} c^\mu x^\nu - \lambda) \gamma. \quad (2.16)$$

This shows that the surface of the projective hypercone defined in Eq. (2.10) behaves as the six-dimensional space-time we started with. In this way, we have derived the conformal coordinate transformation in Eq. (2.4) from a higher-dimensional perspective, a viewpoint that will be useful later in this paper as well. The eight-dimensional formulation also suggests a way of formulating manifestly conformally invariant quantities, an observation that hopefully will be pursued in future work.

2.2 Conformal field transformations

Having discussed the conformal group and its action on the six-dimensional space-time thoroughly, we now turn to the question of how the conformal group generators act on fields defined over this space-time. This action consists of two parts: one due to the dependence of the field in question on the space-time coordinate x^μ and one due to specific properties of the field itself. The latter part consists of the action of the stability subgroup of $x = 0$ (the little group) on the fields, e.g. the transformation properties of a vector field under Lorentz transformations. We will (in contrast to our convention in the previous subsection) adopt the *active* view on conformal transformations, meaning that the space-time coordinates are fixed but the fields change upon such a transformation. By considering the stability subgroup mentioned before, it can be found [13] that a general field $\phi^i(x)$ (where i denotes some index or indices) transforms according to

$$\begin{aligned} \delta_C \phi^i(x) = & a^\mu \partial_\mu \phi^i(x) + \omega^{\mu\nu} \left[x_\nu \partial_\mu \phi^i(x) + (\Sigma_{\mu\nu} \phi)^i(x) \right] + \\ & + \lambda \left[x^\mu \partial_\mu \phi^i(x) + w \phi^i(x) \right] + c^\mu \left[x^2 \partial_\mu \phi^i(x) - 2x_\mu x^\nu \partial_\nu \phi^i(x) + \right. \\ & \left. + 4x^\nu (\Sigma_{\mu\nu} \phi)^i(x) - 2x_\mu w \phi^i(x) + (k_\mu \phi)^i(x) \right], \end{aligned} \quad (2.17)$$

where $\Sigma_{\mu\nu}$, w and k_μ are defined through

$$\begin{aligned} [M_{\mu\nu}, \phi^i(0)] &= (\Sigma_{\mu\nu} \phi)^i(0) \\ [D, \phi^i(0)] &= w \phi^i(0) \\ [K_\mu, \phi^i(0)] &= (k_\mu \phi)^i(0). \end{aligned} \quad (2.18)$$

Here, $M_{\mu\nu}$, D and K_μ should not be confused with the differential expressions in the table above; in these equations (when acting on fields), they denote the full generators of Lorentz transformations, dilatations and special conformal transformations, respectively.

Thus, $\Sigma_{\mu\nu}$ and k_μ correspond to the intrinsic properties of the field under space-time rotations and special conformal transformations, respectively, while w is the conformal weight of the field. Together, these generate the stability subgroup of $x = 0$. We also note that a *primary* field corresponds to one having $k_\mu = 0$.

It is convenient to summarize the transformation (2.17) in the expression

$$\delta_C \phi^i(x) = \xi^\mu(x) \partial_\mu \phi^i(x) + \Omega^{\mu\nu}(x) (\Sigma_{\mu\nu} \phi)^i(x) + \Lambda(x) w \phi^i(x) + c^\mu (k_\mu \phi)^i(x), \quad (2.19)$$

where the space-time dependent parameter functions are defined by

$$\xi^\mu(x) \equiv a^\mu + \omega^{\mu\nu}x_\nu + \lambda x^\mu + c^\mu x^2 - 2c \cdot x x^\mu \quad (2.20)$$

$$\Omega^{\mu\nu}(x) \equiv \omega^{\mu\nu} + 4c^{[\mu}x^{\nu]} \quad (2.21)$$

$$\Lambda(x) \equiv \lambda - 2c \cdot x. \quad (2.22)$$

This expression is also stated in [14] and will prove to be useful when dealing with superfields and the superconformal group later in this paper. Note that the expression for $\xi^\mu(x)$ in Eq. (2.20) coincides with the expression for δx^μ in Eq. (2.4), indicating that this part corresponds to the change in the field due to its dependence on the space-time coordinates.

When acting on *fields*, the generators obey the same commutation relations as in Eq. (2.7), but with the opposite sign; this is due to the difference between active and passive transformations.

To conclude, the conformal transformation of a general field is specified by stating its conformal weight (w), its behaviour under Lorentz transformations ($\Sigma_{\mu\nu}$) and its properties under special conformal transformations (k_μ).

2.3 The bosonic tensor multiplet

The next step is to apply the results of the previous section to the fields in the bosonic sector of the $(2,0)$ tensor multiplet, consisting of five scalar fields and a two-form gauge potential. We denote the scalar fields, transforming in the vector representation of the $SO(5)$ *R*-symmetry group, by the antisymmetric matrix ϕ^{ab} , $a, b = 1, \dots, 4$, satisfying the algebraic condition

$$\Omega_{ab}\phi^{ab} = 0, \quad (2.23)$$

where Ω_{ab} is the $SO(5)$ invariant antisymmetric tensor. Obviously, a and b are $SO(5)$ spinor indices. The two-form potential is denoted b and has an associated field strength $h = db$, both *R*-symmetry scalars. Actually, it is only the self-dual part of the field strength that is part of the tensor multiplet, but in order to give a Lagrangian description of the theory [15], we include the anti self-dual part as a spectator field. It is essential to keep this part decoupled when adding interactions to the theory. The field content, along with the reality properties, is more thoroughly discussed in [16].

The fields have the following properties under conformal transformations:

$$\begin{aligned} (\Sigma_{\mu\nu}\phi^{ab}) &= 0 & (k_\mu\phi^{ab}) &= 0 & w_\phi &= 2 \\ (\Sigma_{\mu\nu}b)_{\rho\sigma} &= \eta_{\mu[\rho}b_{\sigma]\nu} - \eta_{\nu[\rho}b_{\sigma]\mu} & (k_\mu b)_{\rho\sigma} &= 0 & w_b &= 2. \end{aligned} \quad (2.24)$$

The conformal weights w_ϕ and w_b are bounded from below by a unitarity condition [17, 18], and due to the BPS property of the tensor multiplet representation, this bound is saturated. The weights coincide with the mass dimensions of the fields.

Using the expression (2.19), this means that the fields of the bosonic theory transform according to

$$\delta_C \phi^{ab}(x) = \xi^\mu(x) \partial_\mu \phi^{ab}(x) + 2\Lambda(x) \phi^{ab}(x) \quad (2.25)$$

$$\delta_C b_{\mu\nu}(x) = \xi^\rho(x) \partial_\rho b_{\mu\nu}(x) + 2\Omega_{[\mu}{}^\rho(x) b_{\nu]\rho}(x) + 2\Lambda(x) b_{\mu\nu}(x) \quad (2.26)$$

$$\delta_C h_{\mu\nu\rho}(x) = \xi^\sigma(x) \partial_\sigma h_{\mu\nu\rho}(x) - 3\Omega_{[\mu}{}^\sigma(x) h_{\nu]\rho}\sigma(x) + 3\Lambda(x) h_{\mu\nu\rho}(x), \quad (2.27)$$

where we have required the conformal transformation operator to commute with the exterior derivative, in the sense that

$$d(\delta_C b) = \delta_C(db) = \delta_C h. \quad (2.28)$$

Note that this conformal transformation operator is defined in an abstract sense, acting in different ways on different fields. Specifically, its explicit form is different when acting on the derivative of a field than on the field itself. The relation (2.28) may seem evident here, but it is useful to relate to it when we discuss the superconformal transformations later in this paper.

Comparing Eqs. (2.26) and (2.27), we see that both b and its exterior derivative $h = db$ are primary fields, meaning that they have no k_μ piece in their transformation laws, cf. Eq. (2.19). It should be noted that, in general, derivatives of primary fields are not necessarily primary, consider e.g. the variation

$$\delta_C (\partial_\mu \phi^{ab}(x)) = \partial_\mu (\delta_C \phi^{ab}(x)) \quad (2.29)$$

$$= \xi^\nu(x) \partial_\nu \partial_\mu \phi^{ab}(x) - \Omega_\mu{}^\nu(x) \partial_\nu \phi^{ab}(x) + \\ + 3\Lambda(x) \partial_\mu \phi^{ab}(x) - 4c_\mu \phi^{ab}(x), \quad (2.30)$$

where we see that the generators of the conformal group act as expected on a field with a subscript vector index and mass dimension 3, apart from the non-primary piece with parameter c_μ . Thus, although ϕ^{ab} is a primary field, its derivative is not.

The bosonic model also includes self-dual strings, described by an embedding field X^μ , which is a function of the world-sheet coordinates (labelled σ^i , $i = 1, 2$) and transforms as a Lorentz vector. In particular, since we have adopted the active viewpoint on conformal transformations, it is natural to conclude that X^μ transforms according to

$$\delta_C X^\mu = -\xi^\mu(X) = -a^\mu - \omega_\nu^\mu X^\nu - \lambda X^\mu - c^\mu X^2 + 2c \cdot X X^\mu, \quad (2.31)$$

i.e. in the same way as the space-time coordinate x^μ in Eq. (2.4), but with the opposite sign. This expression does not look as nice as the transformations of the tensor multiplet fields found above, but if we instead consider the variation of the differential $\tilde{d}X^\mu \equiv d\sigma^i \partial_i X^\mu$ (which is what we will need in explicit calculations), we get that

$$\delta_C(\tilde{d}X^\mu) = -\Omega_\nu^\mu(X) \tilde{d}X^\nu - \Lambda(X) \tilde{d}X^\mu, \quad (2.32)$$

in accordance with the expected transformation of a field with a vector index, having conformal weight $w_{\tilde{d}X} = -1$. Note also, from Eqs. (2.19) and (2.31), that all terms

apart from those involving $\Sigma_{\mu\nu}$, w and k_μ vanish manifestly when transforming the pull-back of a space-time field to the world-sheet of the string.

Using the transformation rules found above, it is a straight-forward task to prove that the interaction described in [9] is conformally invariant. It should be stressed that this model, because of its electromagnetic coupling, suffers from a classical anomaly [3]. This anomaly is cancelled when fermionic degrees of freedom are included in the model in a supersymmetric way.

3 Conformal invariance in the supersymmetric theory

In this section we turn to the complete theory, incorporating fermionic degrees of freedom in the model. We take advantage of the superfield notation in order to simplify expressions and keep them similar and comparable to the bosonic results in the previous section. This section mixes results that have appeared elsewhere in the literature with new findings. It should also be mentioned that, in some cases, our derivation methods are quite different from those that can be found in previous papers on this subject.

3.1 Including spinors in the model

In this subsection, we introduce the notations and conventions used for spinors and state the *bosonic* conformal transformations of the *fermionic* tensor multiplet fields. It summarizes old results, but is included for completeness to keep the paper self-contained.

The $(2,0)$ tensor multiplet includes, apart from the scalars and the two-form potential mentioned in the previous section, four chiral spinors transforming in the spinor representation of the $SO(5)$ R -symmetry group. These are denoted as ψ_α^a where $\alpha = 1, \dots, 4$ is an $SO(5,1)$ Weyl spinor index. An anti-Weyl spinor is denoted by a superscript α index. The spinor fields obey a symplectic Majorana reality condition [19].

It is convenient to use spinor indices also for the bosonic representations of the Lorentz group. This means that we let (by contracting with the appropriate gamma matrices)

$$x^\mu \rightarrow x^{\alpha\beta} = x^{[\alpha\beta]} \quad (3.1)$$

$$\partial_\mu \rightarrow \partial_{\alpha\beta} = \partial_{[\alpha\beta]} \quad (3.2)$$

$$\eta_{\mu\nu} \rightarrow \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} \quad (3.3)$$

$$b_{\mu\nu} \rightarrow b_\alpha^\beta, \text{ such that } b_\alpha^\alpha = 0 \quad (3.4)$$

$$h_{\mu\nu\rho} + (*h)_{\mu\nu\rho} \rightarrow h_{\alpha\beta} = h_{(\alpha\beta)} \quad (3.5)$$

$$h_{\mu\nu\rho} - (*h)_{\mu\nu\rho} \rightarrow h^{\alpha\beta} = h^{(\alpha\beta)}. \quad (3.6)$$

Note that the last two equations conveniently separate the self-dual and anti self-dual parts of the three-form h into different representations of the Lorentz group.

As is indicated by Eq. (3.3), pairs of antisymmetric indices may be raised and lowered according to

$$\partial^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}. \quad (3.7)$$

Finally, we introduce the dot product between vectors in the same way as before, such that

$$\partial \cdot x \equiv \partial_{\alpha\beta}x^{\alpha\beta} = 6. \quad (3.8)$$

After these preliminaries, let us return to the conformal group and its generators. Naively, the bosonic differential generators of the table in Section 2.1 are translated into

generator	parameter
$P_{\alpha\beta} = \partial_{\alpha\beta}$	$a^{\alpha\beta}$
$M_{\alpha\beta;\gamma\delta} = \frac{1}{2}(x_{\gamma\delta}\partial_{\alpha\beta} - x_{\alpha\beta}\partial_{\gamma\delta})$	$\omega^{\alpha\beta;\gamma\delta}$
$D = x^{\alpha\beta}\partial_{\alpha\beta}$	λ
$K^{\alpha\beta} = x^2\partial^{\alpha\beta} - 2x^{\alpha\beta}x^{\gamma\delta}\partial_{\gamma\delta}$	$c_{\alpha\beta},$

but it is more convenient to introduce a dual notation for the Lorentz generator M and its corresponding parameter ω , according to

$$M_{\alpha}{}^{\beta} = \epsilon^{\beta\gamma\delta\varepsilon}M_{\alpha\gamma;\delta\varepsilon} \quad (3.9)$$

$$\omega_{\alpha}{}^{\beta} = \frac{1}{2}\epsilon_{\alpha\gamma\delta\varepsilon}\omega^{\beta\gamma;\delta\varepsilon}, \quad (3.10)$$

in analogy with the notation for the two-form b in Eq. (3.4). This yields that

$$\omega_{\beta}{}^{\alpha}M_{\alpha}{}^{\beta} = \omega^{\alpha\beta;\gamma\delta}M_{\alpha\beta;\gamma\delta}, \quad (3.11)$$

and the differential operator becomes

$$M_{\alpha}{}^{\beta} = x^{\beta\gamma}\partial_{\alpha\gamma} - x_{\alpha\gamma}\partial^{\beta\gamma} = 2x^{\beta\gamma}\partial_{\alpha\gamma} - \frac{1}{2}\delta_{\alpha}{}^{\beta}x \cdot \partial, \quad (3.12)$$

which is traceless as required.

Under a conformal transformation, the spinor field ψ_{α}^a transforms according to

$$\delta_C\psi_{\alpha}^a = \xi^{\gamma\delta}(x)\partial_{\gamma\delta}\psi_{\alpha}^a + \Omega_{\alpha}{}^{\gamma}(x)\psi_{\gamma}^a + \frac{5}{2}\Lambda(x)\psi_{\alpha}^a, \quad (3.13)$$

where the x -dependent parameter functions are the obvious translations from Section 2.1 and become

$$\xi^{\alpha\beta}(x) = a^{\alpha\beta} + \omega_{\gamma}{}^{\alpha}x^{\gamma\beta} + \omega_{\gamma}{}^{\beta}x^{\alpha\gamma} + (\lambda - 2c \cdot x)x^{\alpha\beta} + c^{\alpha\beta}x^2 \quad (3.14)$$

$$\Omega_{\alpha}{}^{\beta}(x) = \omega_{\alpha}{}^{\beta} - 4c_{\alpha\gamma}x^{\beta\gamma} + c \cdot x\delta_{\alpha}{}^{\beta} \quad (3.15)$$

$$\Lambda(x) = \lambda - 2c \cdot x, \quad (3.16)$$

in terms of which the transformations (2.25) and (2.27) are

$$\delta_C \phi^{ab} = \xi^{\gamma\delta}(x) \partial_{\gamma\delta} \phi^{ab} + 2\Lambda(x) \phi^{ab} \quad (3.17)$$

$$\delta_C h_{\alpha\beta} = \xi^{\gamma\delta}(x) \partial_{\gamma\delta} h_{\alpha\beta} + \Omega_\alpha{}^\gamma(x) h_{\gamma\beta} + \Omega_\beta{}^\gamma(x) h_{\alpha\gamma} + 3\Lambda(x) h_{\alpha\beta}. \quad (3.18)$$

This ends the discussion concerning the conformal group with only bosonic generators. In the next subsection, we extend the model to incorporate fermionic generators, thereby forming the full superconformal group.

3.2 Superconformal coordinate transformations

Since we want to treat a supersymmetric model, it is convenient to supplement the six bosonic space-time coordinates $x^{\alpha\beta}$ by a set of sixteen fermionic coordinates θ_a^α , see [10] for a more thorough introduction of these concepts. The fermionic coordinates are anticommuting, i.e. Grassmann odd, as usual and transform as anti-Weyl spinors under Lorentz rotations and as spinors under R -symmetry transformations.

Following the logic of the previous section, the obvious next step is to find out what the generators of the conformal group $SO(6, 2)$ look like in this *superspace* and how they act on the coordinates. Evidently, the generators must change in comparison to the bosonic case, in order to incorporate the non-trivial action of the conformal group on the fermionic coordinates θ_a^α . For example, since θ_a^α transforms as an anti-Weyl spinor under Lorentz rotations, the corresponding generator $(M_\alpha{}^\beta)$ must be modified. The commutation relations in Eq. (2.7) are expected to remain unchanged as expressed in terms of abstract generators.

However, we want to go a bit further and also include the R -symmetry group $SO(5)$, generated by $U^{ab} = U^{(ab)}$, and supersymmetry, which is generated by the fermionic Q_α^a . We are then forced to introduce the generators S_a^α of *special supersymmetry* as well, which arise as the commutator of the supersymmetry and the special conformal symmetry generators. Altogether, we have then arrived at the superconformal group $OSp(8^*|4)$, which is the expected symmetry group of the complete theory. We will denote the parameters of supersymmetry, R -symmetry and special supersymmetry transformations by η_a^α , v_{ab} and ρ_α^a , respectively. Note that Q_α^a , S_a^α , η_a^α and ρ_α^a all are fermionic (Grassmann odd) quantities.

In this subsection, we will content ourselves with the action of the superconformal group on the coordinates in superspace, postponing their action on fields to the next subsection. Starting from the purely bosonic parts of the generators of $SO(6, 2)$ ($P_{\alpha\beta}$, $M_\alpha{}^\beta$, D and $K^{\alpha\beta}$) obtained above and the well-known generator of supersymmetry (Q_α^a), we can make suitable ansätze for the unknown but essential additional parts (including fermionic variables and derivatives) of these generators as well as for the newly introduced generators (S_a^α and U^{ab}). By requiring these generators to obey certain of the commutation relations of the $OSp(8^*|4)$ group (which is related to the requirement that the algebra should close), the unknown coefficients may be determined and the

resulting differential generators are found to be

$$P_{\alpha\beta} = \partial_{\alpha\beta} \quad (3.19)$$

$$M_\alpha^\beta = 2x^{\beta\gamma}\partial_{\alpha\gamma} - \frac{1}{2}\delta_\alpha^\beta x \cdot \partial + \theta_c^\beta \partial_\alpha^c - \frac{1}{4}\delta_\alpha^\beta \theta \cdot \partial \quad (3.20)$$

$$D = x \cdot \partial + \frac{1}{2}\theta \cdot \partial \quad (3.21)$$

$$\begin{aligned} K^{\alpha\beta} = & -4x^{\alpha\gamma}x^{\beta\delta}\partial_{\gamma\delta} - \Omega^{ab}\Omega^{cd}\theta_a^\gamma\theta_b^\delta\theta_c^\alpha\theta_d^\beta + \\ & + 2\theta_c^{[\alpha}(2x^{\beta]\gamma} - i\Omega^{ab}\theta_a^{\beta]}\theta_b^\gamma)\partial_\gamma^c \end{aligned} \quad (3.22)$$

$$Q_\alpha^a = \partial_\alpha^a - i\Omega^{ac}\theta_c^\gamma\partial_{\alpha\gamma} \quad (3.23)$$

$$\begin{aligned} S_a^\alpha = & \Omega_{ac}(2x^{\alpha\gamma} - i\Omega^{bd}\theta_b^\alpha\theta_d^\gamma)\partial_\gamma^c + 2i\theta_a^\gamma\theta_c^\alpha\partial_\gamma^c - \\ & - i\theta_a^\gamma(2x^{\delta\alpha} - i\Omega^{bd}\theta_b^\delta\theta_d^\alpha)\partial_{\gamma\delta} \end{aligned} \quad (3.24)$$

$$U^{ab} = \frac{1}{2}(\Omega^{ac}\theta_c^\gamma\partial_\gamma^b + \Omega^{bc}\theta_c^\gamma\partial_\gamma^a), \quad (3.25)$$

where the dot product between two fermionic quantities a_a^α and b_β^b is defined by

$$a \cdot b \equiv a_c^\gamma b_\gamma^c. \quad (3.26)$$

Note that the fermions involved in this scalar product are required to have opposite $SO(5, 1)$ chirality, i.e. one should be a Weyl spinor and the other an anti-Weyl spinor.

The generators of the superconformal group act on the superspace coordinates $x^{\alpha\beta}$ and θ_a^α in the usual way; the resulting transformation laws are (similar transformations but in a different notation, and derived in a different way, also appear in [20])

$$\begin{aligned} \delta x^{\alpha\beta} = & a^{\alpha\beta} + \omega_\gamma^\alpha x^{\gamma\beta} + \omega_\gamma^\beta x^{\alpha\gamma} + \lambda x^{\alpha\beta} + 4c_{\gamma\delta}x^{\gamma\alpha}x^{\beta\delta} - i\Omega^{ab}\eta_a^{[\alpha}\theta_b^{\beta]} + \\ & + c_{\gamma\delta}\Omega^{ac}\Omega^{bd}\theta_a^{[\alpha}\theta_b^{\beta]}\theta_c^\gamma\theta_d^\delta - 2i\rho_\gamma^c\theta_c^{[\alpha}x^{\beta]\gamma} - \rho_\gamma^c\Omega^{ab}\theta_c^{[\alpha}\theta_b^{\beta]}\theta_\gamma^\gamma \end{aligned} \quad (3.27)$$

$$\begin{aligned} \delta\theta_a^\alpha = & (\omega_\gamma^\alpha - 4c_{\gamma\delta}x^{\alpha\delta} - 2ic_{\gamma\delta}\Omega^{cd}\theta_c^\alpha\theta_d^\delta + 2i\rho_\gamma^c\theta_c^\alpha)\theta_a^\gamma + \frac{1}{2}\lambda\theta_a^\alpha + \\ & + \eta_a^\alpha + 2\Omega_{ca}\rho_\gamma^c x^{\gamma\alpha} + i\Omega_{ac}\Omega^{bd}\rho_\gamma^c\theta_b^\gamma\theta_d^\alpha + v_{ac}\Omega^{cd}\theta_d^\alpha, \end{aligned} \quad (3.28)$$

which, as in the bosonic case, look rather messy and complicated. However, it is easily shown that the action on the superspace differentials $e^{\alpha\beta} = dx^{\alpha\beta} + i\Omega^{ab}\theta_a^{[\alpha}d\theta_b^{\beta]}$ and $d\theta_a^\alpha$ can be written as

$$\delta e^{\alpha\beta} = \Omega_\gamma^\alpha(x, \theta)e^{\gamma\beta} + \Omega_\gamma^\beta(x, \theta)e^{\alpha\gamma} + \Lambda(x, \theta)e^{\alpha\beta} \quad (3.29)$$

$$\delta(d\theta_a^\alpha) = \Omega_\gamma^\alpha(x, \theta)d\theta_a^\gamma + \frac{1}{2}\Lambda(x, \theta)d\theta_a^\alpha + V_a^c(\theta)d\theta_c^\alpha + 2\Xi_{\gamma,a}(\theta)e^{\alpha\gamma}, \quad (3.30)$$

where the superspace-dependent parameter functions

$$\Omega_\alpha^\gamma(x, \theta) = \omega_\alpha^\gamma - 4c_{\alpha\delta}x^{\gamma\delta} + c \cdot x\delta_\alpha^\gamma - 2ic_{\alpha\delta}\Omega^{cd}\theta_c^\gamma\theta_d^\delta + 2i\rho_\alpha^c\theta_c^\gamma - \frac{i}{2}\delta_\alpha^\gamma\rho \cdot \theta \quad (3.31)$$

$$\Lambda(x, \theta) = \lambda - 2c \cdot x + i\rho \cdot \theta \quad (3.32)$$

are extensions of Eqs. (3.15) and (3.16) to the superconformal case, while the functions

$$V_d^a(\theta) = -\Omega^{ac}v_{cd} + 4i\Omega^{ac}c_{\gamma\delta}\theta_c^\gamma\theta_d^\delta - 2i\rho_\gamma^a\theta_d^\gamma + 2i\Omega^{ae}\rho_\gamma^f\theta_e^\gamma\Omega_{fd} \quad (3.33)$$

$$\Xi_{\alpha,a}(\theta) = 2c_{\alpha\beta}\theta_a^\beta + \Omega_{ab}\rho_\alpha^b \quad (3.34)$$

are new. Naturally, Ω_α^β and V_b^a are traceless, i.e. they obey $\Omega_\alpha^\alpha = V_a^a = 0$. To the best of our knowledge, this way of presenting the superconformal transformations has not appeared previously in the literature. The advantages of using superspace-dependent parameter functions will be made clearer when we consider transformations of *superfields* in Section 3.3.

The transformations (3.29) and (3.30) contain the expected Lorentz, dilatation and R -symmetry parts (with generalized superspace-dependent parameters), but also a term connecting the variation of $d\theta_a^\alpha$ to $e^{\alpha\beta}$ with the parameter function $\Xi_{\alpha,a}(\theta)$, containing the (constant) parameters $c_{\alpha\beta}$ and ρ_α^a . This separates special conformal and special supersymmetry transformations from the other transformations, which yield no such possibilities. This issue will be further discussed later in this paper and is a superspace analogue to the non-trivial piece k_μ in Eq. (2.19).

It is also interesting to note that the infinitesimal supersymmetric interval length is preserved up to a superspace-dependent scale factor under superconformal transformations, i.e.

$$\delta \left(\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}e^{\alpha\beta}e^{\gamma\delta} \right) = \Lambda(x,\theta)\epsilon_{\alpha\beta\gamma\delta}e^{\alpha\beta}e^{\gamma\delta}. \quad (3.35)$$

This relation may in fact be seen as a definition of the superconformal transformations, in analogy with Eq. (2.3) in the bosonic case.

Continuing along the path taken in Section 2, we calculate the commutation relations for the differential generators in Eqs. (3.19)–(3.25). The result is (similar relations

appear e.g. in [14, 20])

$$\begin{aligned}
[P_{\alpha\beta}, M_\gamma^\delta] &= -2\delta_{[\alpha}^\delta P_{\beta]\gamma} - \tfrac{1}{2}\delta_\gamma^\delta P_{\alpha\beta} & [P_{\alpha\beta}, U^{ab}] &= 0 \\
[K^{\alpha\beta}, M_\gamma^\delta] &= 2\delta_\gamma^{[\alpha} K^{\beta]\delta} + \tfrac{1}{2}\delta_\gamma^\delta K^{\alpha\beta} & [K^{\alpha\beta}, U^{ab}] &= 0 \\
[U^{ab}, D] &= 0 & [M_\alpha^\beta, U^{ab}] &= 0 \\
[U^{ab}, U^{cd}] &= -\Omega^{a(c}U^{d)b} - \Omega^{b(c}U^{d)a} & [P_{\alpha\beta}, D] &= P_{\alpha\beta} \\
[M_\alpha^\beta, D] &= 0 & [K^{\alpha\beta}, D] &= -K^{\alpha\beta} \\
[M_\alpha^\beta, Q_\gamma^a] &= -\delta_\gamma^\beta Q_\alpha^a + \tfrac{1}{4}\delta_\alpha^\beta Q_\gamma^a & [Q_\alpha^a, D] &= \tfrac{1}{2}Q_\alpha^a \\
[M_\alpha^\beta, S_a^\gamma] &= \delta_\alpha^\gamma S_a^\beta - \tfrac{1}{4}\delta_\alpha^\beta S_a^\gamma & [S_a^\alpha, D] &= -\tfrac{1}{2}S_a^\alpha \\
[K^{\alpha\beta}, Q_\gamma^a] &= -2\Omega^{ac}\delta_\gamma^{[\alpha} S_c^{\beta]} & [P_{\alpha\beta}, Q_\gamma^c] &= 0 \\
[P_{\alpha\beta}, S_a^\gamma] &= 2\Omega^{ac}\delta_{[\alpha}^\gamma Q_{\beta]}^c & [K^{\alpha\beta}, S_a^\gamma] &= 0 \\
[M_\alpha^\beta, M_\gamma^\delta] &= \delta_\alpha^\delta M_\gamma^\beta - \delta_\gamma^\beta M_\alpha^\delta & [U^{ab}, Q_\gamma^c] &= \Omega^{c(a}Q_\gamma^{b)} \\
\{Q_\alpha^a, Q_\beta^b\} &= -2i\Omega^{ab}P_{\alpha\beta} & [U^{ab}, S_c^\gamma] &= \delta_c^{(a}\Omega^{b)d}S_d^\gamma \\
\{S_a^\alpha, S_b^\beta\} &= -2i\Omega_{ab}K^{\alpha\beta} \\
\{Q_\alpha^a, S_b^\beta\} &= i\delta_\alpha^\beta \left(\delta_a^b D - 4\Omega_{bc}U^{ac} \right) + 2i\delta_a^b M_\alpha^\beta \\
[P_{\alpha\beta}, K^{\gamma\delta}] &= -4\delta_{[\alpha}^{[\gamma} M_{\beta]}^{\delta]} - 2\delta_{[\alpha}^\gamma \delta_{\beta]}^\delta D,
\end{aligned} \tag{3.36}$$

which together define the superconformal group $OSp(8^*|4)$. It should be noted that these are the relations that apply when the differential operators act on each other, *not* when the generators act on fields. In the latter case, the sign on the right hand side of every relation should be changed. It is straightforward to verify that the super-Jacobi identities are satisfied.

In the same way as we compactified the notation in the bosonic case in Eq. (2.9) by extending space-time with one extra space-like and one extra time-like dimension, we may define new generators $J_{\hat{\alpha}\hat{\beta}}$, where $\hat{\alpha}, \hat{\beta} = 1, \dots, 8$ are chiral spinor indices in *eight* dimensions. The hatted indices decompose into subscript and superscript indices in six dimensions, corresponding to Weyl ($\hat{\alpha} = 1, \dots, 4$) and anti-Weyl ($\hat{\alpha} = 5, \dots, 8$) spinors, respectively. We let

$$\begin{aligned}
J_{\alpha\beta} &= \tfrac{1}{2}P_{\alpha\beta} \\
J_\alpha^\beta &= \tfrac{1}{2}M_\alpha^\beta + \tfrac{1}{4}\delta_\alpha^\beta D = -J_\alpha^\beta \\
J^{\alpha\beta} &= -\tfrac{1}{2}K^{\alpha\beta},
\end{aligned} \tag{3.37}$$

and also define new supercharges $\hat{Q}_{\hat{\alpha}}^a$ according to

$$\begin{aligned}
\hat{Q}_{\hat{\alpha}}^a &= Q_\alpha^a \\
\hat{Q}^{a,\alpha} &= \Omega^{ab}S_b^\alpha,
\end{aligned} \tag{3.38}$$

together forming a chiral spinor in eight dimensions.

These generators, together with the unaltered R -symmetry generator U^{ab} , obey the commutation relations

$$\begin{aligned}\{\hat{Q}_{\hat{\alpha}}^a, \hat{Q}_{\hat{\beta}}^b\} &= -4i(\Omega^{ab}J_{\hat{\alpha}\hat{\beta}} + I_{\hat{\alpha}\hat{\beta}}U^{ab}) & [J_{\hat{\alpha}\hat{\beta}}, U^{ab}] &= 0 \\ [J_{\hat{\alpha}\hat{\beta}}, J_{\hat{\gamma}\hat{\delta}}] &= -I_{\hat{\alpha}[\hat{\gamma}}J_{\hat{\delta}]\hat{\beta}} + I_{\hat{\beta}[\hat{\gamma}}J_{\hat{\delta}]\hat{\alpha}} & [J_{\hat{\alpha}\hat{\beta}}, \hat{Q}_{\hat{\gamma}}^a] &= I_{\hat{\gamma}[\hat{\alpha}}\hat{Q}_{\hat{\beta}]^a} \\ [U^{ab}, U^{cd}] &= -\Omega^{a(c}U^{d)b} - \Omega^{b(c}U^{d)a} & [U^{ab}, \hat{Q}_{\hat{\alpha}}^c] &= \Omega^{c(a}\hat{Q}_{\hat{\alpha}}^{b)},\end{aligned}\quad (3.39)$$

where the symmetric matrix $I_{\hat{\alpha}\hat{\beta}}$ has components

$$\begin{aligned}I_{\alpha\beta} &= 0 \\ I_{\alpha}^{\beta} &= \delta_{\alpha}^{\beta} = I^{\beta}_{\alpha} \\ I^{\alpha\beta} &= 0;\end{aligned}\quad (3.40)$$

it transforms in the singlet representation of $SO(6, 2)$. The commutation relations (3.39) contain all the information of Eq. (3.36), but in a much more compact notation.

It is possible to compactify the notation further by considering the generators in a superspace with eight bosonic and four fermionic dimensions. This yields the $OSp(8^*|4)$ structure in a manifest way; this notation appears in [14, 21]. Introduce a matrix J_{AB} , where $A = (\hat{\alpha}, a)$ and $B = (\hat{\beta}, b)$ are $OSp(8^*|4)$ indices. Note that A and B are superindices, meaning that J_{AB} is symmetric if both indices are fermionic, otherwise it is antisymmetric. We denote this in the standard way by

$$J_{AB} = -(-1)^{AB}J_{BA}. \quad (3.41)$$

J_{AB} contains the different generators of the superconformal group, explicitly we take

$$\begin{aligned}J_{\hat{\alpha}\hat{\beta}} &= J_{\hat{\alpha}\hat{\beta}} \\ J_{\hat{\alpha}}^b &= \frac{i}{2\sqrt{2}}\hat{Q}_{\hat{\alpha}}^b \\ J_{\hat{\beta}}^a &= -\frac{i}{2\sqrt{2}}\hat{Q}_{\hat{\beta}}^a \\ J^{ab} &= iU^{ab}.\end{aligned}\quad (3.42)$$

These generators obey the (anti)commutation relations

$$[J_{AB}, J_{CD}] = -\frac{1}{2}\left(I_{BC}J_{AD} - (-1)^{AB}I_{AC}J_{BD} - (-1)^{CD}I_{BD}J_{AC} + (-1)^{AB+CD}I_{AD}J_{BC}\right) \quad (3.43)$$

where the bracket in the left hand side is an anticommutator if both entries in it are fermionic, otherwise it is a commutator. The superspace metric $I_{AB} = (-1)^{AB}I_{BA}$ is defined by

$$I_{AB} = \begin{pmatrix} 0 & \delta_{\alpha}^{\beta} & 0 \\ \delta_{\beta}^{\alpha} & 0 & 0 \\ 0 & 0 & i\Omega^{ab} \end{pmatrix}. \quad (3.44)$$

In order to make the relation

$$I_{AB}I^{BC} = \delta_A^C \quad (3.45)$$

valid (which is essential if want to raise and lower indices), we also need to define the inverse superspace metric as

$$I^{AB} = \begin{pmatrix} 0 & \delta^\alpha_\beta & 0 \\ \delta_\alpha^\beta & 0 & 0 \\ 0 & 0 & -i\Omega_{ab} \end{pmatrix}. \quad (3.46)$$

Note the resemblance between the (anti)commutation relations in Eq. (3.43) and the well-known Lorentz group commutation relations. This suggests that the superconformal transformations act linearly in a superspace with eight bosonic and four fermionic dimensions.

In Section 2.1, we found the conformal transformations of the bosonic coordinates x^μ in an indirect way, by looking at a projective hypercone embedded in an eight-dimensional space with two time-like directions. In this higher-dimensional space, the conformal group acts linearly. Guided by the $OSp(8^*|4)$ covariant notation introduced above, we would like to perform a similar analysis in the superconformal case.

Let the coordinates in superspace be y_A and introduce a projective supercone by the equation

$$I^{AB}y_Ay_B = 0. \quad (3.47)$$

We will parametrize the supercone in a more implicit manner than we did when considering the hypercone in Section 2.1. Consider a point on the supercone, with coordinates $y_A = (y_\alpha, y^\alpha, y^a)$. It is always possible to introduce a fermionic field $\theta_a^\alpha(y)$ such that for any point on the supercone,

$$y^a = \sqrt{2}\Omega^{ab}\theta_b^\beta y_\beta. \quad (3.48)$$

By requiring θ_a^α to transform as an anti-Weyl spinor under $SO(5,1)$, this field is well-defined in all points on the supercone. In the same manner, we introduce the bosonic field $x^{\alpha\beta}(y) = -x^{\beta\alpha}(y)$ such that

$$y^\alpha = \left(2x^{\alpha\beta} - i\Omega^{ab}\theta_a^\alpha\theta_b^\beta\right)y_\beta \quad (3.49)$$

for any point on the supercone.

It is easily verified that all points y_A of this form lie on the supercone defined by Eq. (3.47). Obviously, we may always multiply $x^{\alpha\beta}$ or θ_a^α by a constant and still remain on the supercone. This explains the notion *projective* supercone.

The next step is to vary these coordinates. The transformations are generated by

$$J_{AB} = -y_{[A}\partial_{B]} = (-1)^{AB}y_{[B}\partial_{A]}, \quad (3.50)$$

which satisfies the commutation relations (3.43), given that

$$\partial_A y_B = I_{AB}. \quad (3.51)$$

The coordinates y_A transform according to

$$\delta y_A = \pi^{CD} J_{CD} y_A = (-1)^{C(A+D)} I_{AC} \pi^{CD} y_D, \quad (3.52)$$

where the parameter matrix is given by

$$\pi^{AB} = \begin{pmatrix} 2a^{\alpha\beta} & \omega_\beta^\alpha + \frac{1}{2}\lambda\delta_\beta^\alpha & -i\sqrt{2}\eta_b^\alpha \\ -\omega_\alpha^\beta - \frac{1}{2}\lambda\delta_\alpha^\beta & -2c_{\alpha\beta} & -i\sqrt{2}\rho_\alpha^c\Omega_{cb} \\ i\sqrt{2}\eta_a^\beta & i\sqrt{2}\rho_\beta^c\Omega_{ca} & -iv_{ab} \end{pmatrix}, \quad (3.53)$$

chosen such that the relation

$$\pi^{AB}J_{AB} = \omega_\alpha^\beta M_\beta^\alpha + a^{\alpha\beta}P_{\alpha\beta} + c_{\alpha\beta}K^{\alpha\beta} + \lambda D + \eta_a^\alpha Q_a^\alpha + \rho_\alpha^a S_a^\alpha + v_{ab}U^{ab} \quad (3.54)$$

is valid.

Since Eq. (3.47) is invariant under a transformation of this type, we may require the left-hand and the right-hand sides of Eqs. (3.48)–(3.49) to transform equally. The implicated transformation properties of the fields $x^{\alpha\beta}(y)$ and $\theta_a^\alpha(y)$ when the y -coordinates are transformed in this way are found to agree exactly with the superconformal transformations of the coordinates $x^{\alpha\beta}$ and θ_a^α in our original superspace (with six bosonic and sixteen fermionic dimensions) in Eqs. (3.27) and (3.28) above! This explains the choice of notation and implies that the rather complicated transformation laws for $x^{\alpha\beta}$ and θ_a^α are a mere consequence of a simple rotation in a superspace with eight bosonic and four fermionic dimensions!

This way of introducing the superspace coordinates and deriving their transformation properties has, as far as we know, not appeared previously in the literature. Presumably, the fact that this works points to some underlying structure of the $(2,0)$ superspace in six dimensions, the nature of which is not clear at the moment.

3.3 Superfields

Having introduced a superspace (in the remainder of this paper, we will work in the usual $(2,0)$ superspace with six bosonic and sixteen fermionic dimensions), the next step is to populate it with superfields. The superfield formulation for the $(2,0)$ tensor multiplet first appeared in [22]; a thorough description of its use in our model can be found in [10]. In this paper, we will content ourselves with a short description of the key aspects.

Define a superfield $\Phi^{ab} = \Phi^{ab}(x, \theta)$, obeying the algebraic constraint

$$\Omega_{ab}\Phi^{ab} = 0 \quad (3.55)$$

and the differential constraint

$$D_\alpha^a\Phi^{bc} + \frac{2}{5}\Omega_{de}D_\alpha^d\left(\Omega^{ab}\Phi^{ec} + \Omega^{ca}\Phi^{eb} + \frac{1}{2}\Omega^{bc}\Phi^{ea}\right) = 0, \quad (3.56)$$

where D_α^a is the covariant superspace derivative, defined according to

$$D_\alpha^a = \partial_\alpha^a + i\Omega^{ac}\theta_c^\gamma\partial_{\alpha\gamma}. \quad (3.57)$$

It is important to note [10] that the differential constraint (3.56) implies that the lowest component of the superfield must obey the free equations of motion for a massless scalar field, i.e. the Klein-Gordon equation. This reflects the fact that we are dealing with an *on-shell* superfield formulation.

It is convenient to define supplementary superfields according to

$$\Psi_\alpha^a = -\frac{2i}{5}\Omega_{bc}D_\alpha^b\Phi^{ca} \quad (3.58)$$

$$H_{\alpha\beta} = \frac{1}{4}\Omega_{ab}D_\alpha^a\Psi_\beta^b, \quad (3.59)$$

but it should be noted that these contain no new degrees of freedom compared to Φ^{ab} .

By definition, a superfield transforms according to

$$\delta_Q\Phi^{ab} = [\eta \cdot Q, \Phi^{ab}] \quad (3.60)$$

under a supersymmetry transformation. Working out this commutator, with Q as in Eq. (3.23), it can be shown (by comparing with the explicit transformations in [16]) that the lowest components of the superfields Φ^{ab} , Ψ_α^a and $H_{\alpha\beta}$ are the tensor multiplet fields ϕ^{ab} , ψ_α^a and $h_{\alpha\beta}$, hence the choice of notation. The differential constraint (3.56) yields the usual free equations of motion for these component fields.

The purpose of this section is to find how the rest of the superconformal transformations act on the superfields. The transformations will, as in the bosonic case, contain one piece including the differential expressions in Eqs. (3.19)–(3.25) and some non-differential pieces. The latter may be derived by requiring that the transformation of Φ^{ab} must satisfy the differential constraint (3.56) when Φ^{ab} itself does. We also require the abstract transformation operator to commute with the covariant derivative in superspace (in the same way as transformations commute with derivatives in the bosonic case, see Eq. (2.28) and the discussion thereafter). Note that this approach of course requires the superfields to be on shell. The transformation of $(2,0)$ superfields was also discussed in [23] using a geometric approach, realizing the transformations as derivations in superspace. We will be more explicit and algebraic in our treatment of the problem, trying to take advantage of the superfield formulation. It is our goal to write the transformations of the superfields in a form similar to the one used in the bosonic case in e.g. Eq. (3.17), inspired by the transformation properties of the superspace differentials in Eqs. (3.29)–(3.30).

After some quite involved computations, it is found that the superfields transform according to (we are still in the active picture, where we transform the fields rather

than the coordinates)

$$\begin{aligned}\delta_C \Phi^{ab} &= \xi^{\gamma\delta}(x, \theta) \partial_{\gamma\delta} \Phi^{ab} + \xi_c^\gamma(x, \theta) \partial_\gamma^c \Phi^{ab} + 2\Lambda(x, \theta) \Phi^{ab} + \\ &\quad + V_d^a(\theta) \Phi^{db} + V_d^b(\theta) \Phi^{ad}\end{aligned}\tag{3.61}$$

$$\begin{aligned}\delta_C \Psi_\alpha^a &= \xi^{\gamma\delta}(x, \theta) \partial_{\gamma\delta} \Psi_\alpha^a + \xi_c^\gamma(x, \theta) \partial_\gamma^c \Psi_\alpha^a + \Omega_\alpha^\gamma(x, \theta) \Psi_\gamma^a + \\ &\quad + \frac{5}{2}\Lambda(x, \theta) \Psi_\alpha^a + V_c^a(\theta) \Psi_\alpha^c - 4\Xi_{\alpha,b}(\theta) \Phi^{ab}\end{aligned}\tag{3.62}$$

$$\begin{aligned}\delta_C H_{\alpha\beta} &= \xi^{\gamma\delta}(x, \theta) \partial_{\gamma\delta} H_{\alpha\beta} + \xi_c^\gamma(x, \theta) \partial_\gamma^c H_{\alpha\beta} + \Omega_\alpha^\gamma(x, \theta) H_{\gamma\beta} + \Omega_\beta^\gamma(x, \theta) H_{\alpha\gamma} + \\ &\quad + 3\Lambda(x, \theta) H_{\alpha\beta} + 3i\Xi_{\alpha,c}(\theta) \Psi_\beta^c + 3i\Xi_{\beta,c}(\theta) \Psi_\alpha^c,\end{aligned}\tag{3.63}$$

where the parameter functions $\Omega_\alpha^\beta(x, \theta)$, $\Lambda(x, \theta)$, $V_b^a(\theta)$ and $\Xi_{\alpha,a}(\theta)$ are those defined in Eqs. (3.31)–(3.34), respectively, while $\xi^{\alpha\beta}(x, \theta) = \delta x^{\alpha\beta}$ and $\xi_a^\alpha(x, \theta) = \delta\theta_a^\alpha$, see Eqs. (3.27) and (3.28). In analogy with the notion of primary fields in the purely bosonic case, we see that the superfield Φ^{ab} is *superprimary* (its transformation does not contain any Ξ -part) while the others are not. Note that the transformations (in this notation) are what one would expect by looking at the indices and mass dimensions of the fields, apart from the parts containing $\Xi_{\alpha,a}(\theta)$, where the numerical coefficients are hard to guess a priori.

From these transformations, the transformation laws for the component fields may be read off. The $SO(6, 2)$ transformations agree, as expected, with Eqs. (3.13), (3.17) and (3.18), while supersymmetry acts according to

$$\begin{aligned}\delta_Q \phi^{ab} &= i\eta_c^\alpha (\Omega^{ac} \psi_\alpha^b - \Omega^{bc} \psi_\alpha^a - \frac{1}{2}\Omega^{ab} \psi_\alpha^c) \\ \delta_Q \psi_\alpha^a &= \Omega^{ab} \eta_b^\beta h_{\alpha\beta} + 2\partial_{\alpha\beta} \phi^{ab} \eta_b^\beta \\ \delta_Q h_{\alpha\beta} &= i\eta_a^\gamma (\partial_{\gamma\alpha} \psi_\beta^a + \partial_{\gamma\beta} \psi_\alpha^a),\end{aligned}\tag{3.64}$$

in agreement with [16]. The special supersymmetry transformations of the component fields are

$$\begin{aligned}\delta_S \phi^{ab} &= -4ix^{\gamma\delta} \rho_\gamma^{[a} \psi_\delta^{b]} - i\Omega^{ab} \Omega_{cd} x^{\gamma\delta} \rho_\gamma^c \psi_\delta^d \\ \delta_S \psi_\alpha^a &= -2\rho_\gamma^a x^{\gamma\delta} h_{\delta\alpha} - 4\rho_\alpha^b \Omega_{bc} \phi^{ca} + 4\rho_\gamma^b \Omega_{bc} x^{\gamma\delta} \partial_{\alpha\delta} \phi^{ac} \\ \delta_S h_{\alpha\beta} &= 2i\Omega_{ab} \rho_\gamma^a x^{\gamma\delta} (\partial_{\delta\alpha} \psi_\beta^b + \partial_{\delta\beta} \psi_\alpha^b) - 3i\Omega_{ab} (\rho_\alpha^a \psi_\beta^b + \rho_\beta^a \psi_\alpha^b),\end{aligned}\tag{3.65}$$

while R -symmetry acts according to

$$\begin{aligned}\delta_U \phi^{ab} &= -\Omega^{ac} v_{cd} \phi^{db} - \Omega^{bc} v_{cd} \phi^{ad} \\ \delta_U \psi_\alpha^a &= -\Omega^{ac} v_{cd} \psi_\alpha^d \\ \delta_U h_{\alpha\beta} &= 0,\end{aligned}\tag{3.66}$$

as suggested by the index structure of the fields.

This completes the analysis of the superconformal transformations of the free tensor multiplet in superspace, but we should also consider the self-dual string. It is, as before,

described by a bosonic embedding field $X^{\alpha\beta}(\sigma)$, but we have to supplement it by a second (fermionic) field $\Theta_a^\alpha(\sigma)$ describing the embedding in the fermionic coordinates. The superconformal transformations of these fields are

$$\begin{aligned}\delta_C X^{\alpha\beta} &= -a^{\alpha\beta} - \omega_\gamma{}^\alpha X^{\gamma\beta} - \omega_\gamma{}^\beta X^{\alpha\gamma} - \lambda X^{\alpha\beta} - 4c_{\gamma\delta} X^{\gamma\alpha} X^{\beta\delta} + i\Omega^{ab} \eta_a^{[\alpha} \Theta_b^{\beta]} - \\ &\quad - c_{\gamma\delta} \Omega^{ac} \Omega^{bd} \Theta_a^{[\alpha} \Theta_b^{\beta]} \Theta_c^\gamma \Theta_d^\delta + 2i\rho_\gamma^c \Theta_c^{[\alpha} X^{\beta]\gamma} + \rho_\gamma^c \Omega^{ab} \Theta_c^{[\alpha} \Theta_a^{\beta]} \Theta_b^\gamma \quad (3.67)\end{aligned}$$

$$\begin{aligned}\delta_C \Theta_a^\alpha &= -(\omega_\gamma{}^\alpha - 4c_{\gamma\delta} X^{\alpha\delta} - 2i\rho_\gamma^c \Omega^{cd} \Theta_c^\alpha \Theta_d^\delta + 2i\rho_\gamma^c \Theta_c^\alpha) \Theta_a^\gamma - \frac{1}{2} \lambda \Theta_a^\alpha - \\ &\quad - \eta_a^\alpha - 2\Omega_{ca} \rho_\gamma^c X^{\gamma\alpha} - i\Omega_{ac} \Omega^{bd} \rho_\gamma^c \Theta_b^\gamma \Theta_d^\alpha - v_{ac} \Omega^{cd} \Theta_d^\alpha, \quad (3.68)\end{aligned}$$

which coincide with Eqs. (3.27) and (3.28) for the corresponding coordinates but with the *opposite sign*, in the same manner as in the purely bosonic model. These variations do not have the same structure as the variations of the superfields Φ^{ab} , Ψ_α^a and $H_{\alpha\beta}$, but if we instead consider the appropriate differentials of these fields, we get something more familiar. The differentials in question are related in an obvious way to the superfield differentials $e^{\alpha\beta}$ and $d\theta_a^\alpha$ and are $E^{\alpha\beta} = \tilde{d}X^{\alpha\beta} + i\Omega^{ab} \Theta_a^{[\alpha} d\Theta_b^{\beta]}$ and $\tilde{d}\Theta_a^\alpha$, where $\tilde{d} = d\sigma^i \partial_i$ denotes a differential operator with respect to the world-sheet parameters σ . The variations of these quantities are

$$\delta_C E^{\alpha\beta} = -\Omega_\gamma{}^\alpha(X, \Theta) E^{\gamma\beta} - \Omega_\gamma{}^\beta(X, \Theta) E^{\alpha\gamma} - \Lambda(X, \Theta) E^{\alpha\beta} \quad (3.69)$$

$$\delta_C (\tilde{d}\Theta_a^\alpha) = -\Omega_\gamma{}^\alpha(X, \Theta) \tilde{d}\Theta_a^\gamma - \frac{1}{2} \Lambda(X, \Theta) \tilde{d}\Theta_a^\alpha - V_a^d(\Theta) \tilde{d}\Theta_d^\alpha - 2\Xi_{\gamma,a}(\Theta) E^{\alpha\gamma}, \quad (3.70)$$

which are similar to Eqs. (3.29) and (3.30) but again with the opposite sign. We see that $E^{\alpha\beta}$ transforms as a superprimary quantity while $\tilde{d}\Theta_a^\alpha$ does not.

In retrospect, what we have done in this section is to generalize the superfield formulation, thereby incorporating the full superconformal group in the formalism. This is a must in order to calculate variations of terms involving superfields under such transformations; using the transformation laws for the component fields would lead us nowhere (we do not even know the explicit expressions for the superfields in terms of component fields to all orders). With the superspace generators, we may in a relatively simple and compact way describe the transformations.

We should also comment on the use of superspace-dependent parameter functions in the transformation laws. This also facilitates the calculations, since Lorentz and R -symmetry covariance is manifest through the use of tensor notation. They must, however, be applied with care since these parameter functions generally do not commute with derivatives. To the best of our knowledge, this approach to the superconformal transformations of $(2, 0)$ superfields has not appeared previously in the literature.

It would be interesting to consider the problem of finding a field theory in the superspace with eight bosonic and four fermionic dimensions discussed at the end of the previous subsection. This theory would, presumably, incorporate manifest superconformal symmetry and probably yield some new insights into the properties of the six-dimensional theory.

3.4 The supersymmetric interaction terms

In this subsection, we consider the theory describing the supersymmetric coupling of a self-dual string to a $(2,0)$ tensor multiplet background [10]. As usual, background coupling means that the tensor multiplet fields are taken to be on shell, i.e. they obey their free equations of motion.

The interaction is described by the action terms

$$S^{\text{int}} = - \int_{\Sigma} d^2\sigma \sqrt{\Phi \cdot \Phi} \sqrt{-G} + \int_D F, \quad (3.71)$$

where the first term is a Nambu-Goto term where the string tension has been replaced by an expression in the superfield Φ^{ab} discussed in the previous subsection. Explicitly, the $SO(5)$ invariant scalar product is defined by

$$\Phi \cdot \Phi \equiv \frac{1}{4} \Omega_{ac} \Omega_{bd} \Phi^{ab} \Phi^{cd}, \quad (3.72)$$

and its appearance can be understood by considering the relation between the moduli parameters and the string tension mentioned in the introduction. Note that the term involves the *pull-back* of the superfield Φ^{ab} to the string world-sheet Σ . In the second factor of the Nambu-Goto term, G denotes the determinant of the induced metric in superspace, which may be written as

$$G = \frac{1}{2} \epsilon^{ik} \epsilon^{jl} G_{ij} G_{kl} \quad (3.73)$$

where

$$G_{ij} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} E_i^{\alpha\beta} E_j^{\gamma\delta}. \quad (3.74)$$

Here $E_i^{\alpha\beta}$ is determined by the relation $E^{\alpha\beta} = d\sigma^i E_i^{\alpha\beta}$. Naturally, Σ denotes the string world-sheet.

The second term in Eq. (3.71) is of Wess-Zumino type and involves the pull-back of a certain super three-form F to the world-volume of a "Dirac membrane" D attached to the string, satisfying $\partial D = \Sigma$. For a more elaborate discussion of the Dirac membrane and its properties, we refer to [10, 24]. D is described, similarly to the string, by embedding fields $X^{\alpha\beta}$ and Θ_a^α , but these fields are functions of three parameters instead of two. As an embedding, they naturally transform in the same way as the string embedding fields, i.e. according to Eqs. (3.67) and (3.68), under a superconformal transformation.

The super three-form F is expressed as

$$\begin{aligned} F = & \frac{1}{6} e^{\gamma_1\gamma_2} \wedge e^{\beta_1\beta_2} \wedge e^{\alpha_1\alpha_2} \epsilon_{\alpha_2\beta_2\gamma_1\gamma_2} H_{\alpha_1\beta_1} - \frac{i}{2} e^{\gamma_1\gamma_2} \wedge e^{\beta_1\beta_2} \wedge d\theta_a^\alpha \epsilon_{\alpha\beta_1\beta_2\gamma_1} \Psi_{\gamma_2}^a + \\ & + \frac{i}{2} e^{\gamma_1\gamma_2} \wedge d\theta_b^\beta \wedge d\theta_a^\alpha \epsilon_{\alpha\beta\gamma_1\gamma_2} \Phi^{ab} \end{aligned} \quad (3.75)$$

in terms of the superfields Φ^{ab} , Ψ_α^a and $H_{\alpha\beta}$. It was used by us in [10] but has also appeared (in a slightly different notation) in [23, 25]. The coefficients in the

expression for F make it a closed form in superspace, i.e. $dF = 0$, where d is the superspace exterior derivative. The last statement is only true if the superfields obey the differential constraint (3.56), but this is always the case on shell.

The relative coefficient in Eq. (3.71) is determined by requiring invariance under a local fermionic κ -symmetry, which also removes half of the degrees of freedom contained in Θ_a^α , as required by supersymmetry [10]. The closure of F is essential for this to work, but it also implies that the choice of Dirac membrane D , given a string world-sheet $\Sigma = \partial D$, should have no physical significance.

Next, let us consider a superconformal variation of the interaction (3.71). The Nambu-Goto term is clearly invariant; this is easily seen from the expressions (3.61) and (3.69) for the variations of Φ^{ab} and $E^{\alpha\beta}$, respectively. Since all fields are super-primary, the only thing we really need to check is that the conformal weights match appropriately.

The variation of the Wess-Zumino term is a bit more involved, mainly because of the terms involving $\Xi_{\alpha,a}(\theta)$ in the transformation laws for $\tilde{d}\Theta_a^\alpha$, Ψ_α^a and $H_{\alpha\beta}$. It turns out that the pull-back of the super three-form F to the Dirac membrane world-volume D is superconformally invariant if and only if the coefficients are chosen as in Eq. (3.75), i.e. with the same choice of coefficients that makes it a closed three-form in superspace! This means that the interaction in Eq. (3.71) indeed is superconformally invariant, but *only* with the specific three-form in Eq. (3.75)! It should also be stressed that, out of all the transformations in the superconformal group, it is only the special conformal and the special supersymmetry transformations that put any restrictions on the coefficients. With another choice, the theory would still be e.g. supersymmetric. The model is therefore an example of a theory where special conformal invariance does not follow immediately from dilatational, translational and Lorentz invariance.

Summing up, we have found that the requirement of κ -symmetry and that of superconformal invariance impose equivalent restrictions on the coefficients of the super three-form F , but in different ways. Note, however, that κ -symmetry also determines the relative coefficient between the Nambu-Goto term and the Wess-Zumino term. The superconformal symmetry of the theory has no influence on that. The remarkable agreement between superconformal invariance and κ -symmetry indicates the uniqueness of the theory: the model is tightly constrained by its symmetries. This is arguably the most important result of this paper.

4 The Poincaré dual to the string world-sheet

In the previous section, we worked with an on-shell tensor multiplet. This implied that the super three-form F in Eq. (3.75) was closed and lead us to the conclusion that the choice of Dirac membrane, given a specific string world-sheet Σ , should have no physical significance.

In this section, we try to relax the requirement that F should be closed, as a step in the process of finding a theory where the tensor multiplet fields need not be on shell. We will start in the bosonic case, where the motivation for these ideas is found, and

then turn to the superconformal model and try to generalize the concepts introduced in the bosonic theory.

4.1 The bosonic model

A free two-form gauge field b with a three-form field strength h obeys the Bianchi identity

$$dh = 0 \quad (4.1)$$

and the equations of motion

$$d * h = 0, \quad (4.2)$$

where $*$ denotes the Hodge duality operator. For a self-dual three-form h , these two equations coincide.

If we want to couple this field electromagnetically to a self-dual string (with both magnetic and electric charge), this is done by changing the right-hand sides of the equations above. Explicitly, we get

$$\begin{aligned} dh &= \delta_\Sigma \\ d * h &= \delta_\Sigma, \end{aligned} \quad (4.3)$$

where δ_Σ denotes a four-form called the Poincaré dual to the string world-sheet Σ . It is defined by the relation

$$\int_\Sigma b \equiv \int_M \delta_\Sigma \wedge b, \quad (4.4)$$

where the left integral is evaluated over the string world-sheet, while the right is over the entire six-dimensional Minkowski space. We also note that the self-dual piece of h , expressed as $h_+ = \frac{1}{2}(h + *h)$, obeys an inhomogeneous equation while the anti self-dual piece $h_- = \frac{1}{2}(h - *h)$ is free and therefore decoupled from the theory, as required. It should be emphasized that Eq. (4.3) does not *imply* self-duality, but it is *consistent* with self-duality.

Explicitly, the Poincaré dual is expressed in terms of the string embedding fields X^μ and the space-time coordinates x^μ according to

$$\delta_\Sigma = \frac{1}{4!} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \int_\Sigma \tilde{d}X^\tau \wedge \tilde{d}X^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(x - X), \quad (4.5)$$

where $\tilde{d} \equiv d\sigma^i \partial_i$ again is the differential with respect to the world-sheet variables σ^i , $i = 1, 2$, and $\delta^{(6)}(x - X)$ is a Dirac delta function in six dimensions. This is somewhat analogous to the coupling of a dyonic relativistic particle to a Maxwell field in four dimensions, in that sense δ_Σ is a generalization of the current four-vector [24]. In our model, we couple to a string and δ_Σ is the dual of a current two-form.

In order for this four-form to be consistent with Eq. (4.3), it must be closed. Applying the exterior derivative to δ_Σ , we get that

$$\begin{aligned} d\delta_\Sigma &= \frac{1}{4!} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\kappa \int_\Sigma \tilde{d}X^\tau \wedge \tilde{d}X^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \partial_\kappa \delta^{(6)}(x - X) = \\ &= -\frac{2}{5!} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\kappa \int_\Sigma \tilde{d}X^\tau \wedge \tilde{d}X^\varepsilon \epsilon_{\kappa\mu\nu\rho\sigma\tau} \partial_\varepsilon \delta^{(6)}(x - X) = \\ &= -\frac{2}{5!} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\kappa \int_\Sigma \tilde{d} \left(\tilde{d}X^\tau \epsilon_{\kappa\mu\nu\rho\sigma\tau} \delta^{(6)}(x - X) \right) = 0, \end{aligned} \quad (4.6)$$

where we used the fact that an antisymmetrization over seven vector indices in six dimensions always is zero. The last equality follows since the integrand is a total derivative and the string world-sheet Σ has no boundary.

Next, we are interested in how these relations and quantities behave under (bosonic) conformal transformations. Consider the variation of the three-form h , which according to Eq. (2.27) equals

$$\delta_C h = \frac{1}{3!} dx^\mu \wedge dx^\nu \wedge dx^\rho \left(\xi^\sigma(x) \partial_\sigma h_{\mu\nu\rho}(x) - 3\Omega_\mu^\sigma(x) h_{\sigma\nu\rho}(x) + 3\Lambda(x) h_{\mu\nu\rho}(x) \right). \quad (4.7)$$

However, if we compare this with Eq. (2.6), we see that

$$\delta_C h = \delta_x h, \quad (4.8)$$

where δ_x denotes a *passive* conformal transformation, i.e. one that acts on the space-time coordinates rather than on the fields. This means that, when dealing with h as a differential form, passive and active transformations yield the same result! We will denote such forms as *primary* differential forms. The statement remains true also when exterior derivatives are applied, i.e. $\delta_C(dh) = \delta_x(dh)$, but also for the Hodge dual of h , which means that $\delta_C(*h) = \delta_x(*h)$.

Considering Eq. (4.3), we note that the left-hand sides transform as primary four-forms, therefore the right-hand sides should transform in the same way. Since δ_Σ is an expression in terms of the embedding field X^μ , we may calculate its active transformation explicitly. Before doing this, we want to rewrite the transformation of $\tilde{d}X^\mu$ in Eq. (2.32) so that all parameter functions are expressed in terms of the space-time coordinates x^μ , rather than in terms of the embedding field X^μ . We find that

$$\begin{aligned} \delta_C \left(\tilde{d}X^\mu \right) &= -\Omega^\mu_\nu(x) \tilde{d}X^\nu - \Lambda(x) \tilde{d}X^\mu - 2c_\nu \tilde{d} \left((x - X)^\mu (x - X)^\nu \right) + \\ &\quad + c^\mu \tilde{d} \left((x - X) \cdot (x - X) \right), \end{aligned} \quad (4.9)$$

while the passive variation of this quantity obviously vanishes. We will also need the relation

$$\delta_C \left(dx^\mu \right) = 0 = \delta_x dx^\mu - \Omega^\mu_\nu(x) dx^\nu - \Lambda(x) dx^\mu. \quad (4.10)$$

Finally, considering the Dirac delta function we find that

$$\delta_C \left(\delta^{(6)}(x - X) \right) = -\delta_C X^\mu \partial_\mu \delta^{(6)}(x - X) = \delta_x \left(\delta^{(6)}(x - X) \right) + 6\Lambda(x) \delta^{(6)}(x - X), \quad (4.11)$$

where we have used the relation

$$x^\mu \partial_\nu \delta^{(6)}(x) = -\delta_\nu^\mu \delta^{(6)}(x). \quad (4.12)$$

Putting all this together, again using the properties of the Dirac delta function, we arrive at the conclusion

$$\delta_C(\delta_\Sigma) = \delta_x(\delta_\Sigma), \quad (4.13)$$

meaning that our expression (4.5) for δ_Σ transforms as a primary four-form. This shows that Eq. (4.3) is a well-defined equation with respect to conformal symmetry.

4.2 The superconformal model

Having investigated the bosonic case, we try to generalize these concepts to superspace. We note that the on-shell super three-form F in Eq. (3.75) is primary, i.e.

$$\delta_C F = \delta_{x,\theta} F, \quad (4.14)$$

where $\delta_{x,\theta}$ obviously denotes a passive conformal transformation in superspace. The validity of this equation is most easily seen from the fact that F yields a superconformally invariant Wess-Zumino term, see Section 3.4.

Let us now try to take F off shell, i.e. relax the requirement $dF = 0$. The equation corresponding to Eq. (4.3) is

$$d\mathcal{F} = \Delta_\Sigma, \quad (4.15)$$

where d is the exterior derivative in superspace and \mathcal{F} is a super three-form, not necessarily equal to F defined in Eq. (3.75). The super four-form Δ_Σ appearing on the right-hand side is supposed to be a generalization of the Poincaré dual δ_Σ . To find such a quantity is non-trivial, since there is no proper analogue of the Poincaré dual in superspace. The best we can hope for is to find a super four-form that reduces to δ_Σ if all fermionic degrees of freedom are removed, and that transforms in a nice way under superconformal transformations.

Guided by our previous experience, we try to formulate this four-form in terms of superfields. The fundamental superfields involving the embedding fields $X^{\alpha\beta}$ and Θ_a^α are

$$s^{\alpha\beta} \equiv x^{\alpha\beta} - X^{\alpha\beta} - i\Omega^{ab}\theta_a^{[\alpha}\Theta_b^{\beta]} \quad (4.16)$$

$$t_a^\alpha \equiv \theta_a^\alpha - \Theta_a^\alpha; \quad (4.17)$$

the superfield properties of these are easily verified. The superfield $s^{\alpha\beta}$ is a vector and we will, when appropriate, use the alternative notation s^μ for this, employing an

$SO(5, 1)$ vector index as in the bosonic case. Some important quantities will be stated twice, using both conventions.

We will also need the differential

$$ds^{\alpha\beta} = dx^{\alpha\beta} - i\Omega^{ab}d\theta_a^{[\alpha}\Theta_b^{\beta]} = e^{\alpha\beta} - i\Omega^{ab}(\theta - \Theta)_a^{[\alpha}d\theta_b^{\beta]}, \quad (4.18)$$

where the second equation follows from the definition of the superspace differential $e^{\alpha\beta}$. Similarly, we may differentiate $s^{\alpha\beta}$ with respect to the parameters defining the string world-sheet, yielding

$$\tilde{ds}^{\alpha\beta} = -\tilde{d}X^{\alpha\beta} - i\Omega^{ab}\theta_a^{[\alpha}\tilde{d}\Theta_b^{\beta]} = -E^{\alpha\beta} - i\Omega^{ab}(\theta - \Theta)_a^{[\alpha}\tilde{d}\Theta_b^{\beta]}, \quad (4.19)$$

where $E^{\alpha\beta}$ as before is the bosonic superspace differential expressed in terms of the embedding fields $X^{\alpha\beta}$ and Θ_a^α .

Guided by Eq. (4.5), we introduce

$$\Delta_\Sigma = \frac{1}{4!} \int_{\Sigma} ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge ds^\sigma \wedge \tilde{ds}^\tau \wedge \tilde{s}^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) = \quad (4.20)$$

$$= \frac{1}{4!} \int_{\Sigma} ds^{\delta_1\delta_2} \wedge ds^{\gamma_1\gamma_2} \wedge ds^{\beta_1\beta_2} \wedge ds^{\alpha_1\alpha_2} \tilde{ds}^{\sigma_1\sigma_2} \wedge \tilde{ds}^{\tau_1\tau_2} \times$$

$$\times \epsilon_{\alpha_1\alpha_2\beta_1\gamma_1} \epsilon_{\gamma_2\delta_1\delta_2\sigma_1} \epsilon_{\sigma_2\tau_1\tau_2\beta_2} \delta^{(6)}(s), \quad (4.21)$$

where the Dirac delta function with a grassmannian argument containing both “body” and “soul” is defined in terms of its Taylor expansion. It is apparent that this candidate for Δ_Σ reduces to the bosonic δ_Σ in Eq. (4.5) if all fermions are put to zero.

The second requirement on this Δ_Σ is that it should be closed, as is seen immediately from Eq. (4.15). The proof for this is similar to the bosonic case in Eq. (4.6), but slightly more complicated since $d\tilde{ds}^\mu = \tilde{d}ds^\mu \neq 0$. However, the changes in the proof are minor and therefore omitted in this text.

We also want to investigate how Δ_Σ transforms under superconformal transformations. As in the bosonic case, we will present the transformations piece by piece. Using the explicit variations of $X^{\alpha\beta}$ and Θ_a^α found above, we find that

$$\delta_C(ds^{\alpha\beta}) = \delta_{x,\theta}ds^{\alpha\beta} - \tilde{\Lambda}ds^{\alpha\beta} + 2\tilde{\Omega}_\gamma^{[\alpha}ds^{\beta]\gamma} + d\chi^{\alpha\beta} \quad (4.22)$$

$$\delta_C(\tilde{ds}^{\alpha\beta}) = \delta_{x,\theta}\tilde{ds}^{\alpha\beta} - \tilde{\Lambda}\tilde{ds}^{\alpha\beta} + 2\tilde{\Omega}_\gamma^{[\alpha}\tilde{ds}^{\beta]\gamma} + \tilde{d}\chi^{\alpha\beta}, \quad (4.23)$$

where we have used the presence of a Dirac delta function in the expression (4.21) to put $s^{\alpha\beta} = 0$. Moreover, $\delta_{x,\theta}$ denotes a passive variation in superspace and

$$\tilde{\Omega}_\gamma^\alpha \equiv \Omega_\gamma^\alpha(x, \theta) + i\Omega^{cd}\Xi_{c,\gamma}(\theta)t_d^\alpha - \frac{i}{4}\Omega^{cd}\Xi_{c,\delta}(\theta)t_d^\delta\delta_\gamma^\alpha \quad (4.24)$$

$$\tilde{\Lambda} \equiv \Lambda(x, \theta) + \frac{i}{2}\Omega^{cd}\Xi_{c,\gamma}(\theta)t_d^\gamma \quad (4.25)$$

$$\chi^{\alpha\beta} \equiv -\Omega^{ab}\Omega^{cd}\Xi_{a,\gamma}(\theta)t_b^{[\alpha}t_c^{\beta]}t_d^\gamma - c_{\gamma\delta}\Omega^{ab}\Omega^{cd}t_a^\gamma t_b^{[\alpha}t_c^{\beta]}t_d^\delta. \quad (4.26)$$

Alternatively, we may write these variations as

$$\delta_C(ds^\mu) = \delta_{x,\theta}ds^\mu - \tilde{\Lambda}ds^\mu - \tilde{\Omega}^\mu{}_\nu ds^\nu + d\chi^\mu \quad (4.27)$$

$$\delta_C(\tilde{d}s^\mu) = \delta_{x,\theta}\tilde{d}s^\mu - \tilde{\Lambda}\tilde{d}s^\mu - \tilde{\Omega}^\mu{}_\nu\tilde{d}s^\nu + \tilde{d}\chi^\mu, \quad (4.28)$$

where we again have introduced a vector index instead of an antisymmetric pair of $SO(5, 1)$ spinor indices. We have also introduced $\tilde{\Omega}^\mu{}_\nu$, the definition of which is apparent from how $\omega_\alpha{}^\beta$ was defined from $\omega^{\mu\nu}$ in Section 3.1.

The Dirac delta function transforms according to

$$\delta_C(\delta^{(6)}(s)) = \delta_{x,\theta}\delta^{(6)}(s) + 6\tilde{\Lambda}\delta^{(6)}(s) + \chi \cdot \partial\delta^{(6)}(s), \quad (4.29)$$

where we as usual have used the properties of the delta function to put $s^{\alpha\beta} = 0$. We have also employed the identity

$$s^{\alpha\gamma}\partial_{\alpha\beta}\delta^{(6)}(s) = -\frac{3}{2}\delta_\beta{}^\gamma\delta^{(6)}(s), \quad (4.30)$$

which is analogous to Eq. (4.12) in the bosonic model.

Putting all this together, using the vector index notation, we find that the superconformal variation of Δ_Σ is

$$\begin{aligned} \delta_C\Delta_\Sigma &= \delta_{x,\theta}\Delta_\Sigma + \frac{1}{3!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge d\chi^\sigma \tilde{d}s^\tau \wedge \tilde{d}s^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) + \\ &\quad + \frac{2}{4!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge ds^\sigma \tilde{d}s^\tau \wedge \tilde{d}\chi^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) + \\ &\quad + \frac{1}{4!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge ds^\sigma \tilde{d}s^\tau \wedge \tilde{d}s^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \chi^\kappa \partial_\kappa \delta^{(6)}(s), \end{aligned} \quad (4.31)$$

which may be rewritten as (our conventions for superderivatives may be found in [10])

$$\begin{aligned} \delta_C\Delta_\Sigma &= \delta_{x,\theta}\Delta_\Sigma + d \left[\frac{1}{3!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \tilde{d}s^\tau \wedge \tilde{d}s^\varepsilon \chi^\sigma \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) \right] - \\ &\quad - \frac{2}{4!} \int_\Sigma \tilde{d} \left[ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge ds^\sigma \tilde{d}s^\tau \chi^\varepsilon \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) \right] + \\ &\quad + \frac{7}{4!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \wedge ds^\sigma \tilde{d}s^\tau \wedge \tilde{d}s^\varepsilon \chi^\kappa \epsilon_{[\mu\nu\rho\sigma\tau\varepsilon]} \partial_\kappa \delta^{(6)}(s). \end{aligned} \quad (4.32)$$

In this expression, the last line vanishes since it is an antisymmetrization over seven vector indices in six dimensions, while the second line is zero since it reduces to a boundary term. We are left with

$$\delta_C\Delta_\Sigma = \delta_{x,\theta}\Delta_\Sigma + d\Lambda_\Sigma, \quad (4.33)$$

where the super three-form Λ_Σ is given by

$$\Lambda_\Sigma = \frac{1}{3!} \int_\Sigma ds^\mu \wedge ds^\nu \wedge ds^\rho \tilde{d}s^\tau \wedge \tilde{d}s^\varepsilon \chi^\sigma \epsilon_{\mu\nu\rho\sigma\tau\varepsilon} \delta^{(6)}(s) = \quad (4.34)$$

$$\begin{aligned} &= \frac{1}{3!} \int_\Sigma ds^{\gamma_1\gamma_2} \wedge ds^{\beta_1\beta_2} \wedge ds^{\alpha_1\alpha_2} \chi^{\varepsilon_1\varepsilon_2} \tilde{d}s^{\sigma_1\sigma_2} \wedge \tilde{d}s^{\tau_1\tau_2} \delta^{(6)}(s) \times \\ &\quad \times \left(\epsilon_{\varepsilon_1\varepsilon_2\beta_1\gamma_1} \epsilon_{\gamma_2\alpha_1\alpha_2\sigma_1} \epsilon_{\sigma_2\tau_1\tau_2\beta_2} - \frac{1}{8} \epsilon_{\varepsilon_1\varepsilon_2\gamma_1\gamma_2} \epsilon_{\sigma_1\sigma_2\alpha_1\alpha_2} \epsilon_{\tau_1\tau_2\beta_1\beta_2} \right), \end{aligned} \quad (4.35)$$

but this quantity is obviously only well-defined modulo an exact three-form.

Looking back at Eq. (4.13) for the bosonic model, we see that the introduction of fermionic degrees of freedom has altered the relation by adding a d -exact term to the right-hand side. This means that our candidate (4.20) for Δ_Σ does not transform exactly as one would expect when comparing with the bosonic theory, i.e. an active and a passive transformation do not yield the same result. We note that Λ_Σ vanishes if all fermions are removed and that it is localized (by means of the Dirac delta function) to the world-sheet of the string.

This means that the simple generalization of the bosonic case that we have tried in this subsection did not work out properly. The most probable reason for this failure is that we are required to add a new ingredient, the matrix $\hat{\phi}^{ab}$, to Δ_Σ . This denotes the vacuum expectation value of the field $\phi^{ab}(x)$, normalized to unit modulus with respect to the scalar product in Eq. (3.72). In other words, $\hat{\phi}^{ab}$ is related to the moduli parameters of the theory, denoting the direction in which R -symmetry is broken by the presence of the tensile string. We have seen the appearance of this quantity previously [16] in the discussion concerning the κ -symmetry of the theory. We will not consider how to formulate a superspace generalization of the Poincaré dual including $\hat{\phi}^{ab}$ in this paper, but we hope to return to the matter in a future publication.

Acknowledgments: I would like to thank Erik Flink and Måns Henningson for stimulating and encouraging discussions.

References

- [1] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978) 149.
- [2] E. Witten, *Some comments on string dynamics*, hep-th/9507121.
- [3] M. Henningsson, *Self-dual strings in six dimensions: Anomalies, the ADE-classification, and the world-sheet WZW-model*, Commun. Math. Phys. **257** (2005) 291–302 [hep-th/0405056].
- [4] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996) 335–350 [hep-th/9510135].
- [5] K. Dasgupta and S. Mukhi, *Orbifolds of M-theory*, Nucl. Phys. **B465** (1996) 399–412 [hep-th/9512196].
- [6] A. Strominger, *Open p-branes*, Phys. Lett. **B383** (1996) 44–47 [hep-th/9512059].
- [7] P. K. Townsend, *D-branes from M-branes*, Phys. Lett. **B373** (1996) 68–75 [hep-th/9512062].
- [8] K. Becker and M. Becker, *Boundaries in M-theory*, Nucl. Phys. **B472** (1996) 221–230 [hep-th/9602071].
- [9] P. Arvidsson, E. Flink and M. Henningsson, *Thomson scattering of chiral tensors and scalars against a self-dual string*, J. High Energy Phys. **12** (2002) 010 [hep-th/0210223].
- [10] P. Arvidsson, E. Flink and M. Henningsson, *Supersymmetric coupling of a self-dual string to a (2,0) tensor multiplet background*, J. High Energy Phys. **11** (2003) 015 [hep-th/0309244].
- [11] P. A. M. Dirac, *Wave equations in conformal space*, Ann. Math. **37** (1936) 429.
- [12] H. A. Kastrup, *Gauge properties of the Minkowski space*, Phys. Rev. **150** (1966) 1183.
- [13] G. Mack and A. Salam, *Finite component field representations of the conformal group*, Ann. Phys. **53** (1969) 174–202.
- [14] P. Claus, R. Kallosh and A. Van Proeyen, *M5-brane and superconformal (0,2) tensor multiplet in 6 dimensions*, Nucl. Phys. **B518** (1998) 117–150 [hep-th/9711161].
- [15] E. Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22** (1997) 103–133 [hep-th/9610234].

- [16] P. Arvidsson, E. Flink and M. Henningsson, *Free tensor multiplets and strings in spontaneously broken six-dimensional (2,0) theory*, J. High Energy Phys. **06** (2003) 039 [[hep-th/0306145](#)].
- [17] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, Adv. Theor. Math. Phys. **2** (1998) 781–846 [[hep-th/9712074](#)].
- [18] V. K. Dobrev, *Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry*, J. Phys. **A35** (2002) 7079–7100 [[hep-th/0201076](#)].
- [19] T. Kugo and P. K. Townsend, *Supersymmetry and the division algebras*, Nucl. Phys. **B221** (1983) 357.
- [20] J.-H. Park, *Superconformal symmetry in six-dimensions and its reduction to four*, Nucl. Phys. **B539** (1999) 599–642 [[hep-th/9807186](#)].
- [21] V. G. Kac, *A sketch of Lie superalgebra theory*, Commun. Math. Phys. **53** (1977) 31–64.
- [22] P. S. Howe, G. Sierra and P. K. Townsend, *Supersymmetry in six dimensions*, Nucl. Phys. **B221** (1983) 331.
- [23] C. Grojean and J. Mourad, *Superconformal 6D (2,0) theories in superspace*, Class. Quant. Grav. **15** (1998) 3397–3409 [[hep-th/9807055](#)].
- [24] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *p -brane dyons and electric-magnetic duality*, Nucl. Phys. **B520** (1998) 179–204 [[hep-th/9712189](#)].
- [25] P. S. Howe, *Aspects of the $D = 6$, (2,0) tensor multiplet*, Phys. Lett. **B503** (2001) 197–204 [[hep-th/0008048](#)].